PERSISTENCE PROBABILITIES FOR STATIONARY INCREMENT PROCESSES

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Abstract. We study the persistence probability for processes with stationary increments. Our results apply to a number of examples: sums of stationary correlated random variables whose scaling limit is fractional Brownian motion; random walks in random sceneries; random processes in Brownian scenery; and the Matheron-de Marsily model in $\mathbb{Z}^2$ with random orientations of the horizontal layers. Using a new approach, strongly related to the study of the range, we obtain an upper bound of optimal order in the general case and improved lower bounds (compared to previous literature) for many processes.

1. Introduction

Persistence concerns the probability that a stochastic process has a long negative excursion. In this paper, we are concerned mainly with discrete-time processes. If $Z = (Z_n)_{n\in\mathbb{N}}$ is a stochastic process, we study the rate of the probability

$$\mathbb{P}\left(\max_{k=1,\dots,T} Z_k \le 1\right), \qquad \text{as } T \to +\infty.$$

In many cases of interest, the above probability decreases polynomially, i.e., as $T^{-\theta+o(1)}$, and it is the first goal to find the persistence exponent θ. For a recent overview on this subject, we refer to the survey [3] and for the relevance in theoretical physics we recommend [23, 7].

The purpose of this paper is to analyse the persistence probability for stationary increment processes, i.e. processes such that for any $k \in \mathbb{N}$, $(Z_{n+k} - Z_k)_{n\in\mathbb{N}} \stackrel{\mathcal{L}}{=} (Z_n)_{n\in\mathbb{N}}$, where $\stackrel{\mathcal{L}}{=}$ means equality in law. Under rather general assumptions, we prove that

$$\mathbb{P}\left(\max_{k=1,\dots,T} Z_k \le 1\right) \approx \mathbb{E}\left[\max_{k=1,\dots,T} Z_k\right]/T \qquad \text{as } T \to +\infty,$$

where $\approx$ means up to a multiplicative term in $T^{o(1)}$ (the multiplicative term is bounded by a constant for the upper bound and, for the lower bound, is larger than a function slowly varying at infinity under additional assumptions). We emphasize the fact that we obtain the exact order when the increments are bounded and that we obtain estimates even if increments admit no exponential moment.

Stationary increments are a feature shared by many stochastic processes that are important in theory and applications, and we shall treat here a number of examples. The first one will be sums of correlated sequences whose scaling limit is fractional Brownian motion. This is the natural analog of random walks, for which persistence probabilities have been studied extensively under the notion of fluctuation theory. For sums of correlated sequences, the persistence exponent is shown to be $1 - H$, where H is the scaling exponent, see Theorem 11 below. As a by-product, we obtain the following improvement for the bounds of the persistence probability of continuous-time fractional Brownian motion (improving [27, 1]).

2000 *Mathematics Subject Classification.* 60F05; 60G52.

Key words and phrases. fractional Brownian motion ; Random walk in random scenery ; persistence .

Theorem 1 (Fractional Brownian motion). *Let $H \in (1/2, 1)$. Let B_H be a fractional Brownian motion of Hurst parameter H. Then, there exists $c > 0$ such that*

$$c^{-1} T^{-(1-H)} (\log T)^{-\frac{1}{2H}} \leq \mathbb{P}\left(\sup_{t \in [0,T]} B_H(t) \leq 1 \right) = O(T^{-(1-H)}).$$

The lower bound holds for any $H \in (0, 1)$.

We remark that the upper bound also holds for $H < 1/2$ with an additional logarithmic factor $(\log T)^{(2-H)/H + o(1)}$, by [1], but we do not improve this result here. However we obtain the bound

$$\mathbb{P}\left(\sup_{t \in [0,T]} B_H(t) \leq -1 \right) = O(T^{-(1-H)})$$

even when $H \leq 1/2$.

A second class of examples is given by random walks in random sceneries (RWRS), and depending on the scaling properties of the random walk and of the environment, respectively, different exponents emerge.

Theorem 2 (Random walk in random scenery). *Let $d \in \mathbb{N}^*$ and let $S = (S_n)_{n \in \mathbb{N}}$ be a random walk on $\mathbb{Z}^d$. If S is recurrent, we assume moreover that $(S_n / n^{1/\alpha})_{n \geq 1}$ converges in distribution for some $\alpha \in [d, 2]$. Let $\xi = (\xi_\ell)_{\ell \in \mathbb{Z}^d}$ be a sequence of independent identically distributed random variables, which are centered and admit moments of every order. Assume that S and ξ are independent. Then,*

$$a_n \, n^{-1+o(1)} \leq \mathbb{P}\left(\max_{m=1,\dots,n} \sum_{k=1}^{m} \xi_{S_k} \leq 1 \right) = O(a_n / n),$$

$$\textit{with} \qquad a_n := \begin{cases} n^{1-\frac{1}{2\alpha}} & \textit{if } \; d = 1 < \alpha \leq 2, \\ \sqrt{n \log n} & \textit{if } \; \alpha = d, \\ \sqrt{n} & \textit{if } \; S \textit{ is transient.} \end{cases}$$

Additionally,

- *if ξ is Gaussian, then there exists $c > 0$ such that*
$$\mathbb{P}\left(\max_{m=1,\dots,n} \sum_{k=1}^{m} \xi_{S_k} \leq 1 \right) \geq c a_n / (n \sqrt{\log n});$$
- *if ξ is bounded, then there exists $c > 0$ such that*
$$\mathbb{P}\left(\max_{m=1,\dots,n} \sum_{k=1}^{m} \xi_{S_k} \leq 1 \right) \geq c a_n / n;$$
- *if $\mathbb{P}(\xi_0 \in \{-1, 0, 1\}) = 1$, then there exists $c > 0$ (specified in Theorem 16) such that*
$$\mathbb{P}\left(\max_{m=1,\dots,n} \sum_{k=1}^{m} \xi_{S_k} \leq -1 \right) \sim c a_n / n, \quad \textit{as } n \to +\infty.$$

For RWRS with integer values, this result is strongly related to the behaviour of the expectation of the range $\mathcal{R}_n$ of the RWRS.

A third class of examples is given by the continuous-time analogues to RWRS, that is random process in Brownian scenery $\Delta_t := \int_{\mathbb{R}} L_t(x) \, dW(x)$, with W a two-sided real Brownian motion and $L_t(x)$ the local time (at position x and at time t) of a stable Lévy process with index

$\alpha \in (1, 2]$ (we also consider more general processes as defined in [12], see Section 5 below). In this context, we obtain the following result (improving [11, 12]).

Theorem 3 (Random process in Brownian scenery)**.** *Under the above assumptions, there exists $c > 0$ such that*

$$c^{-1}\, T^{-\frac{1}{2\alpha}} (\log T)^{-\frac{(1+\alpha)}{(2\alpha-1)}} \leq \mathbb{P}\left(\sup_{t\in[0,T]} \Delta_t \leq 1\right) = O(T^{-\frac{1}{2\alpha}}).$$

The fourth example is the Matheron-de Marsily (MdM) model $M = (M_n)_{n\in\mathbb{N}}$ which has been introduced in [26] (see also [5]) to model fluid transport in a porous stratified medium. M is a nearest neighbour random walk in $\mathbb{Z}^2$, in which each horizontal line has been oriented either to the right or to the left (both with probability 1/2; for a detailed definition see Section 6). The second coordinate $(M_n^{(2)})_{n\in\mathbb{N}}$ of this model is a (lazy) random walk, but the first one $(M_n^{(1)})_{n\in\mathbb{N}}$ is more complicated and has some analogy with a random walk in random scenery. For this process, we obtain the following precise estimate.

Theorem 4 (Persistence probability for the MdM model)**.** *There exists $c > 0$ such that*

$$\mathbb{P}\left(\max_{k=1,\ldots,n} M_k^{(1)} \leq -1\right) \sim cn^{-\frac{1}{4}}, \quad \text{as } n \to +\infty.$$

This result was conjectured by Redner [28] and Majumdar [24]. We mention that the constant c appearing in Theorem 4 is given by an explicit formula involving the Kesten-Spitzer process (see Theorem 19).

The outline of the paper is as follows. The main general result, Theorem 5, is given in Section 2. Its proof is split into various intermediate results, outlined and proved also in Section 2. This general result is then applied to the examples mentioned above: Sums of stationary sequences are treated in Section 3, random walks in random scenery in Section 4, random processes in Brownian scenery in Section 5, and the Matheron-de Marsily model in Section 6.

2. General results for stationary increment processees

Our main general results are contained in the next theorem, in which, as usual, we write $x_- := \max(-x, 0)$. It states that the rate of the persistence probability can be obtained from the scaling of the maximum of the process.

Theorem 5. *Let $(Z_n)_{n\in\mathbb{N}}$ be a centered process with stationary increments such that $Z_0 := 0$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that is regularly varying with exponent $\gamma \in (0, 1)$ such that the following limit exists*

$$B := \lim_{n\to+\infty} a_n^{-1} \mathbb{E}\left[\max_{k=1,\ldots,n} Z_k\right] \in (0, \infty).$$

Then

$$\limsup_{n\to+\infty} \frac{n}{a_n} \mathbb{P}\left(\max_{\ell=1,\ldots,n} Z_\ell \leq -1\right) \leq B.$$

Further, if

(i) *There exists a sub-σ-algebra $\mathcal{F}_0$ such that, given $\mathcal{F}_0$, the increments of Z are positively associated and that their common conditional distribution is independent of $\mathcal{F}_0$,* *then there exists $C > 0$ such that*

$$\forall n \geq 1, \quad \mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell \leq 1\right) \leq C\,\frac{a_n}{n}.$$

Finally, if (i) holds or, alternatively,

(ii) *There exists $p > 1$ such that $\|\max_{k=1,\dots,n} Z_k\|_p = O(a_n)$,*

then

- *If $\mathbb{E}[(Z_1)_-^\beta] < \infty$ for any $\beta > 0$ then*

$$\forall n \geq 1, \quad \mathbb{P}(\max_{\ell=1,\dots,n} Z_\ell < 0) \geq a_n n^{-1+o(1)}.$$

- *If there exists $\varepsilon > 0$ such that $\mathbb{E}[e^{-\varepsilon Z_1}] < \infty$ then there exists $c > 0$ such that*

$$\forall n \geq 1, \quad \mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell < 0\right) \geq c\frac{a_n}{n \log n}.$$

- *If there exists $\varepsilon > 0$ such that $\mathbb{E}[e^{\varepsilon (Z_1)_-^2}] < \infty$ (e.g. if Z_1 is Gaussian) then*

$$\forall n \geq 1, \quad \mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell < 0\right) \geq c\frac{a_n}{n\sqrt{\log n}}.$$

- *If $(Z_1)_-$ is bounded, then*

$$\forall n \geq 1, \quad \mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell < 0\right) \geq c\frac{a_n}{n}.$$

Theorem 5 will appear as a consequence of Theorem 8 and Proposition 9 (for the upper bounds) and Theorem 10 (for the lower bounds) that are given in the next two subsections.

2.1. **Upper bounds for the persistence probability.** As explained in the introduction, we consider a stochastic process $(Z_n)_{n\in\mathbb{N}}$ with stationary increments. We start with a general upper bound that will be useful in the setup where $\mathbb{P}(Z_1 \in \mathbb{Z}) = 1$ and which is strongly related to the study of the range of $(Z_n)_{n\in\mathbb{N}}$.

Lemma 6. *Let $(Z_n)_{n\in\mathbb{N}}$ be a centered process with stationary increments such that $Z_0 := 0$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to $+\infty$ such that*

$$\limsup_{n\to+\infty} a_n^{-1}\mathbb{E}\left[\max_{k=1,\dots,n} Z_k - \min_{k=1,\dots,n} Z_k\right] \leq A.$$

Then

$$\limsup_{n\to+\infty} \frac{n}{a_n}\mathbb{P}\left(\min_{\ell=1,\dots,n} |Z_\ell| \geq 1\right) \leq A. \tag{1}$$

When Z_1 is $\mathbb{Z}$-valued, we consider $T_0 := \inf\{n \geq 1 \ : \ Z_n = 0\}$ the first return time of Z to 0. Let $\mathcal{R}_n$ be the range of Z up to time n, i.e. $\mathcal{R}_n := \#\{Z_0, ..., Z_n\}$.

Theorem 7. *Assume we are in the situation of Lemma 6 and that additionally $\mathbb{P}(Z_1 \in \mathbb{Z}) = 1$. Then*

$$\limsup_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) \leq A. \tag{2}$$

Further, if $(a_n)_{n\in\mathbb{N}}$ is a regularly varying sequence with exponent $\gamma \in (0, 1)$ and if

$$C_- := \liminf_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} > 0\,,$$

and

$$C_+ := \limsup_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} < +\infty,$$

then

$$0 < \liminf_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) \le \limsup_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) < \infty\,. \tag{3}$$

Moreover, if the sequence $\left(\frac{\mathbb{E}[\mathcal{R}_n]}{a_n}\right)_{n\ge 0}$ *converges to* C *as* $n \to +\infty$*, then*

$$\lim_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) = \gamma C. \tag{4}$$

In the particular case where $\mathbb{P}(Z_1 \in \{-1, 0, 1\}) = 1$ *and where* (4) *holds, we obtain that*

$$\mathbb{P}\left(\max_{k=1,\dots,n} Z_k \le -1\right) = \frac{1}{2}\mathbb{P}(T_0 > n) \sim \frac{\gamma\, a_n}{2n} C. \tag{5}$$

The next fact is the crucial part in the upper bound in the general (i.e. not necessarily $\mathbb{Z}$-valued) case.

Theorem 8. *Let* $(Z_n)_{n\in\mathbb{N}}$ *be a centered process with stationary increments such that* $Z_0 := 0$ *and let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers such that*

$$\limsup_{n\to+\infty} a_n^{-1}\mathbb{E}\left[\max_{k=0,\dots,n} Z_k\right] \le B.$$

Then

$$\limsup_{n\to+\infty} \frac{n}{a_n}\mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell \le -1\right) \le B. \tag{6}$$

The next technical proposition allows to pass from the boundary -1 in the last fact to the boundary 1.

Proposition 9. *Assume that there exists a sub-*σ*-algebra* $\mathcal{F}_0$ *such that, given* $\mathcal{F}_0$*, the increments of* Z *are positively associated and that their common conditional distribution is independent of* $\mathcal{F}_0$*. Then, for any* $m > 0$ *such that* $\mathbb{P}(Z_1 \le -m) > 0$*,*

$$\mathbb{P}(\max_{\ell=1,\dots,n} Z_\ell \le -m) \le \mathbb{P}(\max_{\ell=1,\dots,n} Z_\ell \le 1) \le \frac{\mathbb{P}(\max_{\ell=1,\dots,n} Z_\ell \le -m)}{(\mathbb{P}(Z_1 \le -m))^{\lfloor 1/m\rfloor+2}}.$$

2.2. **Lower bounds for the persistence probability.** The lower bound is much more complicated to establish in the general case and will require additional assumptions.

Theorem 10. *Let* $(Z_n)_{n\in\mathbb{N}}$ *be a centered process with stationary increments such that* $Z_0 := 0$*. Assume that* $\varepsilon > 0$ *and that* $(a_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers such that*

$$D_1 := \liminf_{n\to+\infty} a_n^{-1}\mathbb{E}\left[\max_{k=0,\dots,n+\lfloor\varepsilon n\rfloor} Z_k\right] > \limsup_{n\to+\infty} a_n^{-1}\mathbb{E}\left[\max_{k=0,\dots,n} Z_k\right] =: D_2.$$

Assume moreover that $(b_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers for which one of the following assumptions holds true:*

(i) $(Z_n)_{n\in\mathbb{N}}$ *satisfies the assumption of Proposition 9 and*

$$\limsup_{n\to+\infty} n\mathbb{P}(-Z_1 > b_n) < \frac{1}{\varepsilon}\left(1 - \frac{D_2}{D_1}\right).$$

(ii) *There exists $p>1$ such that*

$$\limsup_{n\to+\infty} a_n^{-1}\|\sup_{k=0,\dots,n+\lfloor\varepsilon n\rfloor} Z_k\|_p(\varepsilon n\mathbb{P}(-Z_1>b_n))^{1/q}<D_1-D_2,$$

where q is the conjugate of p (i.e. q is such that $1/p+1/q=1$).

Then

$$\liminf_{n\to+\infty}\frac{nb_n}{a_n}\mathbb{P}\Big(\max_{k=1,\dots,n} Z_k<0\Big)>0.$$

2.3. **Proofs of the intermediate results.**

Proof of Lemma 6. Since Z has stationary increments,

$$\begin{aligned} n\mathbb{P}\left(\min_{\ell=1,\dots,n}|Z_\ell|\geq 1\right) &\leq \sum_{k=1}^n \mathbb{P}\left(\min_{\ell=1,\dots,k}|Z_\ell|\geq 1\right)\\ &= \sum_{k=1}^n \mathbb{P}\left(\min_{\ell=1,\dots,k}|Z_{n-k+\ell}-Z_{n-k}|\geq 1\right)\\ &\leq \sum_{k=1}^n \mathbb{P}\left(\forall\ell=1,...,k,\ \lfloor Z_{n-k+\ell}\rfloor\neq\lfloor Z_{n-k}\rfloor\right)\\ &\leq \mathbb{E}\left[\#\{\lfloor Z_0\rfloor,\dots,\lfloor Z_n\rfloor\}\right]\\ &\leq \mathbb{E}\left[\max_{k=1,\dots,n}\lfloor Z_k\rfloor-\min_{k=1,\dots,n}\lfloor Z_k\rfloor+1\right]\\ &\leq \mathbb{E}\left[\max_{k=1,\dots,n} Z_k-\min_{k=1,\dots,n} Z_k+2\right]. \end{aligned}$$

□

Proof of Theorem 7. Since $\mathbb{P}(Z_1\in\mathbb{Z})=1$,

$$\mathbb{P}(T_0>n)=\mathbb{P}\left(\min_{\ell=1,\dots,n}|Z_\ell|\geq 1\right)$$

and so (2) follows from Lemma 6.

Let us prove (3). Note that since Z_ℓ is $\mathbb{Z}$-valued and using stationary increments, we have

$$\begin{aligned} \mathbb{E}\mathcal{R}_n &= \mathbb{E}\left[\#\{Z_1,\dots,Z_n\}\right]\\ &= \sum_{k=0}^n \mathbb{P}(\forall\ell=1,\dots,k: Z_{n-k+\ell}\neq Z_{n-k})\\ &= \sum_{k=0}^n \mathbb{P}(\forall\ell=1,\dots,k: Z_\ell\neq 0)\\ &= \sum_{k=0}^n \mathbb{P}(T_0>k). \end{aligned} \tag{7}$$

Since $(\mathbb{P}(T_0>k))_k$ is non increasing, for every $0<x<1<y$, we have by (7)

$$\frac{\mathbb{E}[\mathcal{R}_{\lfloor yn\rfloor}-\mathcal{R}_n]}{\lfloor yn\rfloor-n}\leq\mathbb{P}(T_0>n)\leq\frac{\mathbb{E}[\mathcal{R}_n-\mathcal{R}_{\lfloor xn\rfloor}]}{n-\lfloor xn\rfloor}.$$

Hence, writing $C_- := \liminf_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n}$ and $C_+ := \limsup_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n}$, we obtain

$$\frac{y^\gamma C_- - C_+}{y-1} \le \liminf_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) \le \limsup_{n\to+\infty} \frac{n}{a_n}\mathbb{P}(T_0 > n) \le \frac{C_+ - x^\gamma C_-}{1-x}.$$

When $C_+ = C_- = C$, (4) is deduced by letting the variables x and y converge to one.

Assume now that $\mathbb{P}(Z_1 \in \{-1,0,1\}) = 1$. Observe that by centering, the random variable Z_1 must be symmetric. Therefore,

$$\begin{aligned} \mathbb{P}\left(\max_{k=1,\dots,n} Z_k \le -1\right) &= \mathbb{P}(T_0 > n,\ Z_1 < 0) \\ &= \mathbb{P}(T_0 > n,\ Z_1 > 0) \\ &= \frac{1}{2}\mathbb{P}(T_0 > n). \end{aligned}$$

Now, by observing that

$$\mathbb{E}[\mathcal{R}_n] = \mathbb{E}\left[\sup_{k=1,\dots,n} Z_k - \inf_{k=1,\dots,n} Z_k + 1\right] \sim C\, a_n,$$

relation (5) directly follows from (4).

□

Proof of Theorem 8. Let $Z^*_{n,m} := \max_{k=n,\dots,m} Z_k$. Since the increments of Z are stationary and $Z_0 = 0$, we deduce that $Z^*_{k+1,k+n} - Z_k$ has the same distribution as $Z^*_{1,n}$ and so

$$\begin{aligned} n\,\mathbb{P}(Z^*_{1,n} \le -1) &\le \sum_{k=0}^{n-1} \mathbb{P}(Z^*_{1,n-k} \le -1) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(Z^*_{k+1,n} \le Z_k - 1) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(1 + Z^*_{k+1,n} \le Z_k) \\ &\le \mathbb{E}[Z^*_{0,n}], \end{aligned}$$

since $\#\{k = 0, \dots, n-1 \,:\, 1 + Z^*_{k+1,n} \le Z_k\} \le Z^*_{0,n} - Z_n$ and $\mathbb{E}[Z_n] = 0$. □

Proof of Proposition 9. Let $K \in \mathbb{N}$ be such that $Km > 1$, so that $-(K+1)m + 1 \le -m$. Let $n \in \mathbb{N}^*$. Given $\mathcal{F}_0$, the increments of $(Z_n)_{n\in\mathbb{N}}$ are positively associated, hence

$$\begin{aligned} &\mathbb{P}\left(\max_{\ell=1,\dots,n+K+1} Z_\ell \le -m \,\middle|\, \mathcal{F}_0\right) \\ &\quad\ge \mathbb{P}\left(\forall k = 1,\dots,K+1,\ (Z_k - Z_{k-1}) \le -m,\ \max_{\ell=K+2,\dots,n+K+1} (Z_\ell - Z_{K+1}) \le 1 \,\middle|\, \mathcal{F}_0\right) \\ &\quad\ge \left(\prod_{k=1}^{K+1} \mathbb{P}\left(Z_k - Z_{k-1} \le -m \middle| \mathcal{F}_0\right)\right) \mathbb{P}\left(\max_{\ell=K+2,\dots,n+K+1} (Z_\ell - Z_{K+1}) \le 1 \,\middle|\, \mathcal{F}_0\right) \\ &\quad\ge \left(\prod_{k=1}^{K+1} \mathbb{P}(Z_1 \le -m)\right) \mathbb{P}\left(\max_{\ell=K+2,\dots,n+K+1} (Z_\ell - Z_{K+1}) \le 1 \,\middle|\, \mathcal{F}_0\right), \end{aligned}$$

since the conditional distribution of $Z_k - Z_{k-1}$ given $\mathcal{F}_0$ is the distribution of Z_1. So

$$\begin{aligned}\mathbb{P}\left(\max_{\ell=1,\dots,n+K+1} Z_\ell \le -m\right) &\ge (\mathbb{P}(Z_1 \le -m))^{K+1}\,\mathbb{P}\left(\max_{\ell=K+2,\dots,n+K+1}(Z_\ell - Z_{K+1}) \le 1\right)\\ &\ge (\mathbb{P}(Z_1 \le -m))^{K+1}\,\mathbb{P}\left(\max_{\ell=1,\dots,n} Z_\ell \le 1\right),\end{aligned}$$

since the increments of Z are stationary. □

Proof of Theorem 10. Since the increments of Z are stationary and $Z_0 = 0$, we can deduce that $\mathbb{P}(\max_{k=1,\dots,n} Z_k < 0) = \mathbb{P}(Z^*_{k+1,n+k} < Z_k)$.

Let $\varepsilon > 0$ and $M_n := \#\{k = 0, ..., \lfloor \varepsilon n\rfloor - 1 \ : \ Z^*_{k+1,n+\lfloor \varepsilon n\rfloor} < Z_k\}$. Note that

$$\begin{aligned}\lfloor \varepsilon n\rfloor\,\mathbb{P}(Z^*_{1,n} < 0) &\ge \sum_{k=0}^{\lfloor \varepsilon n\rfloor -1} \mathbb{P}\left(Z^*_{1,\lfloor (1+\varepsilon)n\rfloor - k} < 0\right)\\ &= \sum_{k=0}^{\lfloor \varepsilon n\rfloor -1} \mathbb{P}\left(Z^*_{k+1,\lfloor (1+\varepsilon)n\rfloor} < Z_k\right)\\ &= \mathbb{E}\left[M_n\right]. \qquad (8)\end{aligned}$$

We set $R_{M_n+1} := \inf\{k = \lfloor \varepsilon n\rfloor, ..., n+\lfloor \varepsilon n\rfloor \ : \ Z_k > Z^*_{k+1,n+\lfloor \varepsilon n\rfloor}\}$ and

$$\forall i = 1, ..., M_n, \quad R_i := \sup\{k = 0, ..., R_{i+1} - 1 \ : \ Z^*_{k+1,n+\lfloor \varepsilon n\rfloor} < Z_k\}.$$

Note that $Z_{R_1} = Z^*_{0,n+\lfloor \varepsilon n\rfloor}$ and $Z_{R_{M_n+1}} = Z^*_{\lfloor \varepsilon n\rfloor, n+\lfloor \varepsilon n\rfloor}$. Hence

$$\begin{aligned}Z^*_{0,n+\lfloor \varepsilon n\rfloor} - Z^*_{\lfloor \varepsilon n\rfloor,n+\lfloor \varepsilon n\rfloor} &= Z_{R_1} - Z_{R_{M_n+1}} = \sum_{i=1}^{M_n}(Z_{R_i} - Z_{R_{i+1}})\\ &\le \sum_{i=1}^{M_n}(Z_{R_i} - Z_{R_i+1})\\ &\le \left(\sup_{k=1,\dots,\lfloor \varepsilon n\rfloor}(Z_{k-1} - Z_k)\right) M_n,\end{aligned}$$

and so, for $b_n > 0$,

$$\begin{aligned}\mathbb{E}[M_n] &\ge \mathbb{E}\left[M_n \mathbf{1}_{\{\sup_{k=1,\dots,\lfloor \varepsilon n\rfloor}(Z_{k-1}-Z_k)\le b_n\}}\right]\\ &\ge (b_n)^{-1}\,\mathbb{E}\left[(Z^*_{0,n+\lfloor \varepsilon n\rfloor} - Z^*_{\lfloor \varepsilon n\rfloor,n+\lfloor \varepsilon n\rfloor})\mathbf{1}_{\{\sup_{k=1,\dots,\lfloor \varepsilon n\rfloor}(Z_{k-1}-Z_k)\le b_n\}}\right]\\ &\ge (b_n)^{-1}\left(\mathbb{E}\left[Z^*_{0,n+\lfloor \varepsilon n\rfloor}\mathbf{1}_{\{\sup_{k=1,\dots,\lfloor \varepsilon n\rfloor}(Z_{k-1}-Z_k)\le b_n\}}\right] - \mathbb{E}[Z^*_{0,n}]\right),\end{aligned}$$

since $\mathbb{E}[Z^*_{\lfloor \varepsilon n\rfloor,n+\lfloor \varepsilon n\rfloor}] = \mathbb{E}[Z^*_{0,n}]$.

Assumption (i). Then

$$\begin{aligned}\mathbb{E}[M_n] &\ge (b_n)^{-1}\left(\mathbb{E}\left[\mathbb{E}[Z^*_{0,n+\lfloor \varepsilon n\rfloor}|\mathcal{F}_0]\,\mathbb{P}\left(\sup_{k=1,\dots,\lfloor \varepsilon n\rfloor}(Z_{k-1}-Z_k)\le b_n \,\middle|\, \mathcal{F}_0\right)\right] - \mathbb{E}[Z^*_{0,n}]\right)\\ &\ge (b_n)^{-1}\left(\mathbb{E}\left[\mathbb{E}[Z^*_{0,n+\lfloor \varepsilon n\rfloor}|\mathcal{F}_0](1-\varepsilon n\mathbb{P}(-Z_1 > b_n))\right] - \mathbb{E}[Z^*_{0,n}]\right)\\ &= (b_n)^{-1}\left(\mathbb{E}\left[Z^*_{0,n+\lfloor \varepsilon n\rfloor}\right](1-\varepsilon n\mathbb{P}(-Z_1 > b_n)) - \mathbb{E}[Z^*_{0,n}]\right), \qquad (9)\end{aligned}$$

where we used the fact that the increments of Z are positively associated conditionally to $\mathcal{F}_0$, and that their common conditional distribution is independent of $\mathcal{F}_0$. Let η be such that $\limsup_{n\to+\infty} n\mathbb{P}(-Z_1 > b_n) < \eta < \frac{1}{\varepsilon}\left(1 - \frac{D_2}{D_1}\right)$. For n large enough, $\varepsilon n\mathbb{P}(-Z_1 > b_n) \le \varepsilon\eta$. Due to (8) and (9), we obtain

$$\varepsilon n\mathbb{P}(Z^*_{1,n} < 0) \ge (b_n)^{-1}\,\mathbb{E}\left[(1-\varepsilon\eta)Z^*_{0,n+\lfloor\varepsilon n\rfloor} - Z^*_{0,n}\right].$$

Therefore

$$\liminf_{n\to+\infty} \frac{\varepsilon n\, b_n}{a_n}\mathbb{P}(Z^*_{1,n} < 0) \ge (1-\varepsilon\eta)D_1 - D_2 > 0.$$

Assumption (ii). Let q be such that $1/p + 1/q = 1$ and let η be such that

$$\limsup_{n\to+\infty} a_n^{-1}\|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\|_p(\varepsilon n\mathbb{P}(-Z_1 > b_n))^{1/q} < \eta < D_1 - D_2.$$

Then, for n large enough,

$$\begin{aligned}
&\mathbb{E}\left[Z^*_{0,n+\lfloor\varepsilon n\rfloor}\mathbf{1}_{\{\sup_{k=1,\dots,\lfloor\varepsilon n\rfloor}(Z_{k-1}-Z_k)>b_n\}}\right]\\
\le\ & \|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\|_p\left(\mathbb{P}\left(\sup_{k=1,\dots,\lfloor\varepsilon n\rfloor}(Z_{k-1}-Z_k) > b_n\right)\right)^{\frac{1}{q}}\\
\le\ & \|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\|_p\,(\varepsilon n\mathbb{P}(-Z_1 > b_n))^{\frac{1}{q}} \le \eta a_n
\end{aligned}$$

and so

$$\mathbb{E}[M_n] \ge (b_n)^{-1}\left(\mathbb{E}\left[Z^*_{0,n+\lfloor\varepsilon n\rfloor} - Z^*_{0,n}\right] - \eta a_n\right);$$

from which we obtain

$$\liminf_{n\to+\infty} \frac{\varepsilon n\, b_n}{a_n}\mathbb{P}(Z^*_{1,n} < 0) \ge D_1 - D_2 - \eta > 0.$$

□

Proof of Theorem 5. The upper bounds follow from Theorem 8 and Proposition 9.

For the lower bounds, we assume that $\mathbb{E}[G((Z_1)_-)] < \infty$ with either $G(t) = e^{at}$ or $G(t) = t^{\beta'}$ or $G(t) = e^{at^2}$ for some $a > 0$ or some $\beta' > 1$. Due to the Markov inequality,

$$\mathbb{P}(-Z_1 > b_n) = \mathbb{P}((Z_1)_- > b_n) \le \frac{\mathbb{E}[G((Z_1)_-)]}{G(b_n)}.$$

Note that we can apply Theorem 10 with $b_n = c_0\log(n)$ if $G(t) = e^{at}$, $b_n = c_0 n^{\frac{1}{\beta'}}$ if $G(t) = t^{\beta'}$ and $b_n = c_0\sqrt{\log(n)}$ if $G(t) = e^{at^2}$, for a suitable c_0. If $(Z_1)_-$ is bounded, we choose $b_n := \|(Z_1)_-\|_\infty$. □

3. Sums of stationary sequences and fractional Brownian motion

Let $(X_i)_{i\ge 0}$ be a stationary centered Gaussian sequence with variance 1 and correlations $r(j) := \mathbb{E}[X_0X_j] = \mathbb{E}[X_kX_{j+k}]$ satisfying as $n \to +\infty$,

$$\sum_{i,j=1}^{n} r(i-j) = n^{2H}\ell(n), \tag{10}$$

where $H \in (0,1)$ and ℓ is a slowly varying function at infinity. We are interested in the persistence exponent of the sum of the sequence $Z_n := \sum_{i=1}^n X_i$ for $n \ge 1$ and $Z_0 := 0$.

We recall that the scaling limit of $(Z_n)_{n\in\mathbb{N}}$ is the fractional Brownian motion B_H with Hurst parameter H, (see [30], [33, Theorem 4.6.1]):

$$\left(n^{-H}\ell(n)^{-1/2}Z_{[nt]}\right)_{t\geq 0}\overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}}(B_H(t))_{t\geq 0}\,, \tag{11}$$

and that B_H is a real centered Gaussian process with covariance function

$$\mathbb{E}[B_H(t)B_H(s)]=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

A sequence satisfying relation (10) is said to have *long-range dependence* if $H>1/2$. We refer to [29] for a recent overview of the field.

In this setup the following theorem is valid.

Theorem 11. *Assume $(X_i)_{i\geq 0}$ is a stationary centered Gaussian sequence such that (10) holds with $H\in(0,1)$ and ℓ slowly varying. Then there is some constant $c>0$ such that, for every $n\geq 1$,*

$$c^{-1}\;n^{-(1-H)}\frac{\sqrt{\ell(n)}}{\sqrt{\log n}}\leq\mathbb{P}\left(\max_{k=1,\dots,n}Z_k<0\right)\;\text{ and }\mathbb{P}\left(\max_{k=1,\dots,n}Z_k\leq -1\right)\leq c\;n^{-(1-H)}\sqrt{\ell(n)}. \tag{12}$$

If moreover, $\inf_n\sum_{i=1}^n r(i-1)>0$, then there is some constant $c>0$ such that

$$\forall n\geq 1,\quad c^{-1}n^{-(1-H)}\frac{\sqrt{\ell(n)}}{\sqrt{\log n}}\leq\mathbb{P}\left(\max_{k=1,\dots,n}Z_k\leq 1\right)\leq n^{-(1-H)}e^{c\sqrt{\log n}}.$$

If moreover the correlation function r is non-negative (which implies that $H\geq 1/2$) then there is some constant $c>0$ such that

$$\forall n\geq 1,\quad c^{-1}n^{-(1-H)}\frac{\sqrt{\ell(n)}}{\sqrt{\log n}}\leq\mathbb{P}\left(\max_{k=1,\dots,n}Z_k\leq 1\right)\leq c\;n^{-(1-H)}\sqrt{\ell(n)}.$$

Note that (12) gives a lower bound for the boundary 0 and an upper bound for the boundary -1. These estimates will follow more or less directly from Theorem 5. In order to change the boundaries (to boundary $+1$, or in fact to any finite constant), one has to proceed differently according to whether the sequence (X_i) is positively correlated (which is easier) or whether it may also be negatively correlated (in which case we need the extra condition on the sum of the correlations).

Proof of Theorem 11. Assume (10). We set $\sigma_n:=\sup_{k=1,\dots,n}a_k$ where $a_n:=\|Z_n\|_2=n^H\sqrt{\ell(n)}$. Due to Karamata's characterization of slowly varying functions [18], there exists $c_1>0$ and a function ε such that $\lim_{t\to+\infty}\varepsilon(t)=0$ and such that $c_1^{-1}b_n\leq a_n\leq c_1b_n$ with $b_n:=n^He^{\int_0^n\frac{\varepsilon(t)}{t}\,dt}$. Let us denote by D_n the Dudley integral

$$D_n:=\int_0^{\sigma_n}\sqrt{\log N(n,t)}\mathrm{d}t, \tag{13}$$

where $N(n,t)$ is the smallest number of open balls of $\{0,\dots,n\}$ of radius t for the pseudo-metric $d(k,\ell)=a_{|k-\ell|}$ which form a covering of $\{0,\dots,n\}$. Note that if $a_r\leq c_1b_r\leq t$ for some $r<n$ then $N(n,t)\leq 2n/r$. Further, trivially $N(n,t)\leq n+1$ for any t and $N(n,t)=1$ for $t>a_n$.

Therefore, for $\vartheta \in (0,1)$, for n large enough,

$$\begin{aligned} D_n &\leq \int_0^{b_n/\sqrt{\log(n+1)}} \sqrt{\log(n+1)}\,\mathrm{d}t + \int_{b_n/\sqrt{\log(n+1)}}^{c_1 b_n} \sqrt{\log N(n,t)}\,\mathrm{d}t \\ &\leq b_n + \sum_{k=n^\vartheta}^{n} \int_{c_1 b_{k-1}}^{c_1 b_k} \sqrt{\log(2n/(k-1))}\,dt \\ &\leq b_n + \sum_{k=n^\vartheta}^{n} c_1(b_k - b_{k-1})\sqrt{\log(2n/(k-1))}. \end{aligned}$$

But, due to the form of b_k, $\frac{b_k - b_{k-1}}{b_k} = \frac{k^H-(k-1)^H}{k^H} + \frac{(k-1)^H\left(1-e^{\int_k^{k-1}\frac{\varepsilon(t)}{t}\,dt}\right)}{k^H} = O(k^{-1})$. Let $\upsilon \in (0,H)$. There exists $c_2 > 0$ such that

$$\begin{aligned} D_n &\leq b_n + c_2 b_n \frac{1}{n}\sum_{k=n^\vartheta}^{n} \frac{n b_k}{k b_n}\sqrt{\log(2n/(k-1))} \\ &\leq b_n + c_2 b_n \frac{1}{n}\sum_{k=n^\vartheta}^{n} (k/n)^{H-1-\upsilon} = O(a_n)\,, \end{aligned} \tag{14}$$

for n large enough (where we used again the expression of b_n). Due to [22, Corollary 2, p.181],

$$\forall u>0, \quad \mathbb{P}\Big(\sup_{k=1,\dots,n} Z_k > u + 4\sqrt{2}D_n\Big) \leq \mathbb{P}(Z_1 > u/\sigma_n)\,.$$

Therefore,

$$\begin{aligned} \mathbb{E}\Big[\sup_{k=1,\dots,n} Z_k^2\Big] &\leq 32D_n^2 + \int_{32D_n^2}^{\infty} \mathbb{P}\Big(\sup_{k=1,\dots,n}|Z_k| > \sqrt{u}\Big)\,\mathrm{d}u \\ &= 32D_n^2 + 2\int_{4\sqrt{2}D_n}^{\infty} \mathbb{P}\Big(\sup_{k=1,\dots,n} Z_k > x\Big)2x\,\mathrm{d}x \\ &= 32D_n^2 + \int_0^\infty \mathbb{P}\left(Z_1 > u/\sigma_n\right)4(4\sqrt{2}D_n + u)\,\mathrm{d}u \\ &= 32D_n^2 + \sigma_n\int_0^\infty \mathbb{P}\left(Z_1 > v\right)4(4\sqrt{2}D_n + v\sigma_n)\,\mathrm{d}v \\ &\leq 32D_n^2 + C(\sigma_n D_n + \sigma_n^2) = O(a_n^2)\,, \end{aligned}$$

due to (14) and the fact that $\sigma_n \sim a_n$ as n is large.

Thus, we have proved that $\|\max_{k=1,\dots,n} Z_k\|_2 = O(a_n)$, which combined with (11) yields

$$\frac{1}{n^H \ell(n)^{1/2}}\,\mathbb{E}\left[\max_{k=1,\dots,n} Z_k\right] \to \mathbb{E}\left[\sup_{t\in[0,1]} B_H(t)\right] \in (0,\infty).$$

Hence, Theorem 5 implies both the first upper bound and the (last) estimate corresponding to the case when r is non-negative (since in this case the increments of Z are then positively associated).

It remains to prove the upper bound of the second point. We are now going to use Proposition 1.6 in [2]. For this purpose, let us denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the process (Z_k), i.e. the kernel Hilbert space belonging to the kernel $K(n,m) := \mathbb{E}[Z_n Z_m]$, $n,m \in \mathbb{N}$. Assume that $\kappa := \inf_n \sum_{i=1}^n r(i-1) = \inf_n K(1,n) > 0$. Consider the function

$f(n) := 2K(1,n)/\kappa \in \mathcal{H}$. Note that $f(n) \geq 2$ and that

$$\|f\|_{\mathcal{H}}^2 = \langle f, f\rangle_{\mathcal{H}} = \frac{2}{\kappa}\langle K(1,.), f\rangle_{\mathcal{H}} = \frac{2}{\kappa} f(1) = (2/\kappa)^2 K(1,1) = (2/\kappa)^2.$$

Using these two properties and Proposition 1.6 in [2] we can conclude that

$$\begin{aligned} cn^{-(1-H)}\ell(n)^{1/2} &\geq \mathbb{P}(\max_{k\leq n} Z_k \leq -1) \\ &= \mathbb{P}(\forall k \leq n \,:\, Z_k + f(k) \leq -1 + f(k)) \\ &\geq \mathbb{P}(\forall k \leq n \,:\, Z_k + f(k) \leq 1) \\ &\geq \mathbb{P}(\forall k \leq n \,:\, Z_k \leq 1)\, \exp(-\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(\forall k \leq n \,:\, Z_k \leq 1))} - \|f\|_{\mathcal{H}}^2/2) \end{aligned}$$

Inserting now the lower bound given by (12) and the value for $\|f\|_{\mathcal{H}}$ we obtain, for every $\alpha > H - 1$,

$$cn^{-(1-H)}\ell(n)^{1/2} \geq \mathbb{P}(\forall k \leq n \,:\, Z_k \leq 1)\, \exp(-\sqrt{8\alpha\kappa^{-2}\log n} - 2/\kappa^2),$$

for n large enough. This shows the upper bound for the desired probability. □

Now let us finally prove the consequences for the (continuous-time) fractional Brownian motion.

Proof of Theorem 1. A simple example in Theorem 11 is fractional Gaussian noise: If B_H denotes a (continuous-time) fractional Brownian motion, define $X_i := B_H(i+1) - B_H(i)$, $i = 0, 1, \ldots$. Theorem 11 holds and gives that there exists some constant $c > 0$ such that for large enough n

$$\mathbb{P}\left(\max_{k=1,\ldots,n} B_H(k) \leq -1\right) \leq c\, n^{-(1-H)}.$$

When $H \in (\frac{1}{2}, 1)$, we obtain

$$\mathbb{P}\left(\max_{k=1,\ldots,n} B_H(k) \leq 1\right) \leq c\, n^{-(1-H)},$$

while for $H \in (0, \frac{1}{2})$ we get

$$\mathbb{P}\left(\max_{k=1,\ldots,n} B_H(k) \leq 1\right) \leq c\, n^{-(1-H)} e^{c\sqrt{\log n}}.$$

From these computations, we can deduce an alternative proof of the upper bound for the persistence probability of the continuous-time fractional Brownian motion because trivially:

$$\mathbb{P}\left(\sup_{t\in[0,T]} B_H(t) \leq 1\right) \leq \mathbb{P}\left(\max_{k=1,\ldots,[T]} B_H(k) \leq 1\right).$$

The estimate for the lower bound are more involved. Due to Theorem 10, there exists $c > 0$ such that

$$\mathbb{P}\left(\max_{k=1,\ldots,n} B_H(k) \leq 0\right) > c\frac{n^{-(1-H)}}{\sqrt{\log n}}.$$

Moreover,

$$\max_{t\in[0,n]} B_H(t) - \max_{k=1,\ldots,n\lfloor (C\log n)^{1/(2H)}\rfloor} B_H(n_k) \leq \max_{k=1,\ldots,n\lfloor (C\log n)^{1/(2H)}\rfloor} A_k,$$

with

$$A_k := \max_{j\in[n_k,n_{k+1}]} (B_H(j) - B_H(n_k))$$

where $n_k := \frac{k-1}{\lfloor (C\log n)^{1/(2H)}\rfloor}$. Then, using the self-similarity of the fractional Brownian motion,

$$\begin{aligned}\mathbb{P}\left(\max_{k=1,\dots,n\lfloor (C\log n)^{1/(2H)}\rfloor} A_k > 1\right) &\le n(C\log n)^{1/(2H)}\mathbb{P}\left(\max_{t\in[0,1]} B_H(t\lfloor (C\log n)^{1/(2H)}\rfloor^{-1}) > 1\right)\\ &\le n(C\log n)^{1/(2H)}\mathbb{P}\left(\max_{t\in[0,1]} B_H(t) > \lfloor (C\log n)^{1/(2H)}\rfloor^{H}\right)\\ &\le n(C\log n)^{1/(2H)} e^{-aC\log n}\end{aligned}$$

for some constant $a > 0$ (see [27] for instance). We choose C so that this last quantity is in $o(n^{-(1-H)}(\log n)^{-\frac{1}{2H}})$. Then

$$\begin{aligned}&\mathbb{P}\left(\max_{t\in[0,n]} B_H(t) \le 1\right)\\ \ge\ &\mathbb{P}\left(\max_{k=1,\dots,n\lfloor (C\log n)^{1/(2H)}\rfloor} B_H\left(\frac{k}{\lfloor (C\log n)^{1/(2H)}\rfloor}\right) \le 0\right) - o(n^{-(1-H)}(\log n)^{-\frac{1}{2H}})\\ \ge\ &\mathbb{P}\left(\lfloor (C\log n)^{1/2H}\rfloor^{-H} \max_{k=1,\dots,n\lfloor (C\log n)^{1/(2H)}\rfloor} B_H(k) \le 0\right) - o(n^{-(1-H)}(\log n)^{-\frac{1}{2H}})\\ \ge\ &\mathbb{P}\left(\max_{k=1,\dots,n\lfloor (C\log n)^{1/(2H)}\rfloor} B_H(k) \le 0\right) - o(n^{-(1-H)}(\log n)^{-\frac{1}{2H}})\\ \ge\ &cn^{-(1-H)}(\log n)^{-\frac{1}{2H}}.\end{aligned}$$

□

4. Random walks in random sceneries

Random walks in random sceneries were introduced independently by H. Kesten and F. Spitzer [19] and by A. N. Borodin [6]. Let $d \in \mathbb{N}^*$ and $S = (S_n)_{n\in\mathbb{N}}$ be a random walk in $\mathbb{Z}^d$ starting at 0, i.e., $S_0 = 0$ and $X_n := S_n - S_{n-1}, n \ge 1$ is a sequence of i.i.d. $\mathbb{Z}^d$-valued random variables. Let $\xi = (\xi_x)_{x\in\mathbb{Z}^d}$ be a field of i.i.d. real valued random variables independent of S. The field ξ is called the random scenery. *The random walk in random scenery (RWRS)* $Z := (Z_n)_{n\in\mathbb{N}}$ is defined by setting $Z_0 := 0$ and, for $n \in \mathbb{N}^*$, $Z_n := \sum_{i=1}^n \xi_{S_i}$. We will denote by $\mathbb{P}$ the joint law of S and ξ.

Limit theorems for RWRS have a long history, we refer to [16] for a complete review. As in [19], we consider the case when the distribution of ξ_0 is in the normal domain of attraction of a stable distribution of index $\beta \in (1, 2]$ and, if S is recurrent, we assume that the distribution of X_1 is in the normal domain of attraction of a stable distribution of order $\alpha \in [d, 2]$. We assume without any loss of generality that the support of the distribution of X_1 is not contained in a proper subgroup of $\mathbb{Z}^d$ and that the closed subgroup generated by the support of the distribution of ξ_0 is either $\mathbb{Z}$ or $\mathbb{R}$. Under the previous assumptions, the following weak convergence holds in the space $\mathcal{D}([0, +\infty))$ of càdlàg real-valued functions defined on $[0, \infty)$, endowed with the Skorokhod topology (with respect to the classical J_1-metric):

$$\left(n^{-\frac{1}{\alpha}} S_{\lfloor nt\rfloor}\right)_{t\ge0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (U(t))_{t\ge0} \quad \text{if } S \text{ is recurrent}$$

and $\left(n^{-\frac{1}{\beta}} \sum_{k=1}^{\lfloor nt\rfloor} \xi_{\pm k e_1}\right)_{t\ge0} \Longrightarrow^{\mathcal{L}}_{n\to\infty} (Y(\pm t))_{t\ge0}$, where $U(\cdot), Y(\cdot), Y(-\cdot)$ are independent Lévy processes of respective order α, β, β, with $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$. (When $\alpha = d = 2$, the process U corresponds to the two-dimensional Brownian motion). When $d = 1 < \alpha$, in [19], Kesten and

Spitzer proved the following convergence in distribution in $(\mathcal{D}([0,+\infty), J_1)$,

$$(n^{-1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}Z_{[nt]})_{t\geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \left(\Delta_t := \int_{[0,+\infty[} L_t(x)\,\mathrm{d}Y(x) + \int_{[0,+\infty[} L_t(-x)\,\mathrm{d}Y(-x)\right)_{t\geq 0},$$

where $(L_t(x))_{x\in\mathbb{R},t\geq 0}$ is a continuous version with compact support of the local time of the process U (see [25]).

When S is transient, $((n^{-\frac{1}{\beta}}Z_{[nt]})_{t\geq 0})_{n\in\mathbb{N}^*}$ converges in distribution (with respect to the M_1-metric), to $(\Delta_t := c_0 Y(t))_{t\geq 0}$ for some $c_0 > 0$ (see [9]). When $\alpha = d$, $((n^{-\frac{1}{\beta}}(\log n)^{\frac{1}{\beta}-1}Z_{[nt]})_{t\geq 0})_{n\in\mathbb{N}^*}$ converges in distribution (with respect to the M_1-metric), to $(\Delta_t := c_1 Y(t))_{t\geq 0}$ for some $c_1 > 0$ (see [9]). Hence in any of the cases considered above, $((Z_{\lfloor nt\rfloor}/a_n)_{t\geq 0})_{n\in\mathbb{N}}$ converges in distribution (with respect to the M_1-metric) to some process Δ, with

$$a_n := \begin{cases} n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}} & \text{if} \quad d=1,\ \alpha\in(1,2]. \\ n^{\frac{1}{\beta}}(\log n)^{1-\frac{1}{\beta}} & \text{if} \quad \alpha = d. \\ n^{\frac{1}{\beta}} & \text{if} \quad S \text{ is transient.} \end{cases} \tag{15}$$

For every $y\in\mathbb{Z}^d$ and every integer $n\geq 1$, we write $N_n(y)$ for the number of visits of the walk S to site y before time n, i.e.

$$N_n(y) := \#\{k=1,...,n \ : \ S_k = y\}.$$

We also write $R_n := \#\{S_1,...,S_n\}$ for the range of S up to time n. Note that Z and can be rewritten as follows:

$$Z_n = \sum_{y\in\mathbb{Z}^d} \xi_y N_n(y).$$

When Z_1 takes its values in $\mathbb{Z}$, we define the range $\mathcal{R}_n$ of the RWRS Z, i.e. the number of sites visited by Z before time n, by

$$\mathcal{R}_n := \#\{Z_0,\dots,Z_n\}.$$

Remark 12 (Transient RWRS). *Assume that $\beta\in(0,1)$ and that $\mathbb{P}(\xi_1\in\mathbb{Z}) = 1$. Then the RWRS is transient (see for instance* [10]*) and, due to an argument by Derriennic* [34, Lemma 3.3.27]*, $(\mathcal{R}_n/n)_{n\in\mathbb{N}^*}$ converges $\mathbb{P}$-almost surely to $\mathbb{P}[Z_j\neq 0,\ \forall j\geq 1]$.(There, we consider the ergodic dynamical system (Ω,μ,T) given by $\Omega := (\mathbb{Z}^d)^{\mathbb{Z}}\times\mathbb{Z}^{\mathbb{Z}}$, $\mu := (\mathbb{P}_{S_1})^{\otimes\mathbb{Z}}\otimes(\mathbb{P}_{\xi_1})^{\otimes\mathbb{Z}}$ and $T((\alpha_k)_k,(\epsilon_k)_k) := ((\alpha_{k+1})_k,(\epsilon_{k+\alpha_0})_k)$ (see for instance* [17] *for its ergodicity, p. 162). We set $f((\alpha_k)_k,(\epsilon_k)_k) = \epsilon_0$. With these choices, $(Z_j)_{j\geq 1}$ has the same distribution under $\mathbb{P}$ as $(\sum_{k=1}^j f\circ T^k)_{j\geq 1}$ under μ.)*

We assume from now on that $\beta > 1$. Then the RWRS Z is recurrent (see top of the page 2083 in [10]). Let us check that the assumptions of Theorem 5 are satisfied with $(a_n)_{n\in\mathbb{N}}$ given in (15). First, since ξ is a sequence of i.i.d. random variables independent of S, the increments $(\xi_{S_k})_k$ of Z_n are positively associated conditionally to $\mathcal{F}_0$ the σ-algebra generated by S and their conditional distribution given S is the distribution of ξ_0.

Proposition 13. *Assume $\beta > 1$, then*

$$\frac{\max_{k=0,\dots,n} Z_{\lfloor nt\rfloor} - \min_{k=0,\dots,n} Z_{\lfloor nt\rfloor}}{a_n} \xrightarrow{\mathcal{L}} \sup_{t\in[0,1]}\Delta_t - \inf_{t\in[0,1]}\Delta_t$$

and

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\max_{k=0,\dots,n} Z_k\right]}{a_n} = \mathbb{E}\left[\sup_{t\in[0,1]}\Delta_t\right].$$

Proof of Proposition 13. Due to the convergence for the M_1-topology of $((a_n^{-1}Z_{\lfloor nt\rfloor})_t)_{n\in\mathbb{N}^*}$ to $(\Delta_t)_t$ as n goes to infinity, we know that $(a_n^{-1}(\max_{0\le k\le n} Z_k - \min_{0\le \ell\le n} Z_\ell))_{n\in\mathbb{N}^*}$ converges in distribution to $\sup_{t\in[0,1]}\Delta_t - \inf_{s\in[0,1]}\Delta_s$ as n goes to infinity (see Section 12.3 in [33]). Let us prove that $(a_n^{-1}\max_{0\le k\le n} Z_k)_{n\in\mathbb{N}^*}$ is uniformly integrable. To this end we will use the fact that, conditionally to the walk S, the increments of $(Z_n)_{n\in\mathbb{N}}$ are centered and positively associated. Let $\beta'\in(1,\beta)$ be fixed. Due to Theorem 2.1 of [14], there exists some constant $c_{\beta'}>0$ such that

$$\mathbb{E}\left[\left|\max_{j=0,\dots,n} Z_j\right|^{\beta'}|S\right]\le \mathbb{E}\left[\max_{j=0,\dots,n}|Z_j|^{\beta'}|S\right]\le c_{\beta'}\mathbb{E}\left[|Z_n|^{\beta'}|S\right],$$

so $\mathbb{E}\left[|\max_{j=0,\dots,n}Z_j|^{\beta'}\right]\le c_{\beta'}\mathbb{E}\left[|Z_n|^{\beta'}\right]$. It remains to prove that $\mathbb{E}[|Z_n|^{\beta'}]=O(a_n^{\beta'})$.
Let us first consider the easiest case when the random scenery is square integrable that is $\beta=2$, then we take $\beta'=2$ in the above computations and observe that $\mathbb{E}\left[|Z_n|^2\right]=\mathbb{E}[\xi_0^2]\mathbb{E}[V_n]$, where V_n is the number of self-intersections up to time n of the random walk S, i.e. $V_n=\sum_x (N_n(x))^2=\sum_{i,j=1}^n \mathbf{1}_{S_i=S_j}$. Usual computations (see Lemma 2.3 in [4]) give that $\mathbb{E}[V_n]=\sum_{i,j=1}^n\mathbb{P}(S_{i-j}=0)\sim c'(a_n)^2$ and the result follows.
When $\beta\in(1,2)$, let us define $V_n(\beta):=\sum_{y\in\mathbb{Z}}(N_n(y))^\beta$. Given the random walk, Z_n is a sum of independent zero-mean random variables, then from [31, Theorem 3], there exists some constant $C>0$ such that for every n

$$\mathbb{E}[|Z_n|^{\beta'}|S]\le C\sum_y N_n(y)^{\beta'}\mathbb{E}[|\xi_y|^{\beta'}]\le CV_n(\beta'). \tag{16}$$

From which we deduce that $\mathbb{E}[|Z_n|^{\beta'}]\le C\mathbb{E}[V_n(\beta')]$.

If $\alpha>d=1$, due to Lemma 3.3 of [13], we know that $\mathbb{E}[V_n(\beta')]=O\left(a_n^{\beta'}\right)$. In the other cases, using Hölder's inequality, we have $\mathbb{E}[V_n(\beta')]\le \mathbb{E}[R_n]^{1-\frac{\beta'}{2}}\mathbb{E}[V_n]^{\frac{\beta'}{2}}$.

If $\alpha=d$, we know that $\mathbb{E}[R_n]\sim c\frac{n}{\log n}$ (see for instance Theorem 6.9, page 398 in [20]) and $\mathbb{E}[V_n]\sim cn\log n$ so $\mathbb{E}[V_n(\beta')]=O\left(a_n^{\beta'}\right)$ with $a_n=n^{\frac{1}{\beta'}}(\log n)^{1-\frac{1}{\beta'}}$. If S is transient, the expectations of R_n and V_n behaves as n, we deduce that $\mathbb{E}[V_n(\beta')]=O\left(a_n^{\beta'}\right)$ with $a_n=n^{\frac{1}{\beta'}}$. We conclude that

$$\lim_{n\to+\infty}\frac{1}{a_n}\mathbb{E}\left[\max_{j=0,\dots,n}Z_j\right]=\mathbb{E}\left[\sup_{t\in[0,1]}\Delta_t\right].$$

□

Proof of Theorem 2. Using Proposition 13, we apply Theorem 5 with $\mathcal{F}_0$ the σ-algebra generated by S. □

In the case of RWRS, the proof of Theorem 5 can be modified in order to get better lower bounds when ξ has a moment of order $\beta>1$.

Proposition 14. *Assume that ξ has a moment of order $\beta>1$. Then,*

$$\mathbb{P}\Big(\max_{k=1,\dots,n}Z_k\le 1\Big)=\mathcal{O}(a_n/n).$$

- *If $\alpha>1$, then*

$$\liminf_{n\to+\infty} n^{\frac{1}{\alpha}}\,\mathbb{P}\Big(\max_{k=1,\dots,n}Z_k<0\Big)>0.$$

- *If $\alpha = d$, then*
$$\liminf_{n\to+\infty} \frac{n}{\log n}\mathbb{P}\Big(\max_{k=1,\dots,n} Z_k < 0\Big) > 0.$$
- *If $\alpha < 1$, then*
$$\liminf_{n\to+\infty} n\,\mathbb{P}\Big(\max_{k=1,\dots,n} Z_k < 0\Big) > 0.$$

Proof. The upper bound directly follows from Proposition 13 and Theorem 5. Let us prove the three lower bounds. Let $\mathcal{F}_0$ be the σ-algebra generated by S. We use the proof of Theorem 10 and the notations therein. We adapt the proof of Theorem 10 using the fact that

$$\begin{aligned}\mathbb{P}\Big(\sup_{k=1,\dots,\lfloor\varepsilon n\rfloor}(Z_{k-1}-Z_k) > b_n\Big|\mathcal{F}_0\Big) &\le R_{\lfloor\varepsilon n\rfloor}\,\mathbb{P}(-Z_1 > b_n)\\ &\le R_{\lfloor\varepsilon n\rfloor}\,\mathbb{E}[|\xi_0|^\beta]\,b_n^{-\beta}.\end{aligned}$$

Let $\beta' \in (1,\beta)$. Due to (8) and (9),

$$\begin{aligned}\mathbb{P}\Big(Z_0 > Z^*_{1,n}\Big) &\ge \frac{\mathbb{E}[M_n]}{\varepsilon n}\\ &\ge \frac{1}{\varepsilon\, n\, b_n}\Big(\mathbb{E}\Big[Z^*_{0,n+\lfloor\varepsilon n\rfloor} - Z^*_{0,n}\Big] - \mathbb{E}\Big[Z^*_{0,n+\lfloor\varepsilon n\rfloor}R_{\lfloor\varepsilon n\rfloor}\Big]\,\mathbb{P}(-Z_1 > b_n)\Big)\\ &\ge \frac{1}{\varepsilon\, n\, b_n}\left(\mathbb{E}\Big[Z^*_{0,n+\lfloor\varepsilon n\rfloor} - Z^*_{0,n}\Big] - \Big\|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\Big\|_{\beta'}\big\|R_{\lfloor\varepsilon n\rfloor}\big\|_{\beta'/(\beta'-1)}\,\mathbb{E}\Big[|\xi_0|^\beta\Big]b_n^{-\beta}\right).\end{aligned}$$

We have seen in the proof of Proposition 13 that

$$\Big\|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\Big\|_{\beta'} = O(a_n). \tag{17}$$

Moreover

$$\|R_n\|_{\beta'/(\beta'-1)} = O(\mathbb{E}[R_n]). \tag{18}$$

Indeed,

$$\begin{aligned}\mathbb{E}\Big[(R_n/\mathbb{E}[R_n])^p\Big] &= \int_0^{+\infty}\mathbb{P}\Big((R_n/\mathbb{E}[R_n])^p > x\Big)\,dx\\ &= \int_0^{+\infty}\mathbb{P}\Big(R_n > \mathbb{E}[R_n]x^{1/p}\Big)\,dx = \int_0^{+\infty} O\Big(e^{-C(p)x^{1/p}}\Big)\,dx < \infty,\end{aligned}$$

for some $C(p) > 0$ due to [11, Lemma 34]. Now we choose $b_n = O(\mathbb{E}[R_n]^{\frac{1}{\beta}})$ such that

$$\limsup_{n\to+\infty} a_n^{-1}\Big\|Z^*_{0,n+\lfloor\varepsilon n\rfloor}\Big\|_{\beta'}\big\|R_{\lfloor\varepsilon n\rfloor}\big\|_{\beta'/(\beta'-1)}\,\mathbb{E}\Big[|\xi_0|^\beta\Big]b_n^{-\beta} < D_1 - D_2,$$

and so

$$\limsup_{n\to+\infty}\frac{nb_n}{a_n}\mathbb{P}\Big(Z_0 > Z^*_{1,n}\Big) > 0.$$

But $a_n \sim c\,n\mathbb{E}[R_n]^{\frac{1}{\beta}-1}$. Hence $nb_n/a_n = O(\mathbb{E}[R_n])$. □

We are now interesting in the case when the random variables ξ are $\mathbb{Z}$-valued. Better estimations can be obtained in this particular context.

Proposition 15. *Assume $\beta > 1$ and that $\mathbb{P}(\xi_1 \in \mathbb{Z}) = 1$, then*

$$0 < \liminf_{n\to+\infty}\frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \le \limsup_{n\to+\infty}\frac{\mathbb{E}[\mathcal{R}_n]}{a_n} < \infty\,, \tag{19}$$

$$0 < \liminf_{n\to+\infty}\frac{n}{a_n}\mathbb{P}(T_0 > n) \le \limsup_{n\to+\infty}\frac{n}{a_n}\mathbb{P}(T_0 > n) < \infty\,. \tag{20}$$

Proof of the lower bound of Proposition 15. Since $\mathcal{R}_n \le \max_{k=1,\dots,n} Z_k - \min_{k=1,\dots,n} Z_k + 1$, we already now that $\limsup_{n\to+\infty} \mathbb{E}[\mathcal{R}_n]/a_n < \infty$. Due to Theorem 7, it is enough to prove that $\liminf_{n\to+\infty} \mathbb{E}[\mathcal{R}_n]/a_n > 0$. Let $\mathcal{N}_n(x) := \#\{k = 1, ..., n : Z_k = x\}$. Applying the Cauchy-Schwarz inequality to $n = \sum_x \mathcal{N}_n(x)\mathbf{1}_{\{\mathcal{N}_n(x)>0\}}$, we obtain

$$n^2 \le \sum_y \mathbf{1}_{\{\mathcal{N}_n(y)>0\}} \sum_x (\mathcal{N}_n(x))^2 = \mathcal{R}_n \, \mathcal{V}_n,$$

with $\mathcal{V}_n = \sum_x (\mathcal{N}_n(x))^2 = \sum_{i,j=1}^n \mathbf{1}_{\{Z_i=Z_j\}}$ the number of self-intersections of Z up to time n and so using Jensen's inequality, $\frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \ge \frac{n^2}{a_n}\mathbb{E}[(\mathcal{V}_n)^{-1}] \ge \frac{n^2}{a_n}\mathbb{E}[\mathcal{V}_n]^{-1}$. Moreover, using the local limit theorems for the RWRS proved in [10, 9],

$$\mathbb{E}[\mathcal{V}_n] = n + 2 \sum_{1\le i<j\le n} \mathbb{P}(Z_{j-i} = 0) \sim C' \frac{n^2}{a_n}.$$

Hence $\liminf_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \ge \frac{1}{C'} > 0$. □

In the particular case where $\mathbb{P}(\xi_0 \in \{-1, 0, 1\}) = 1$, we obtain a precise estimate (as a consequence of the second part of Theorem 7).

Theorem 16. *Assume that* $\mathbb{P}(\xi_1 \in \{-1, 0, 1\}) = 1$ *(*$\beta = 2$*), then*

$$\frac{\mathcal{R}_n}{a_n} \xrightarrow{\mathcal{L}} \sup_{t\in[0,1]} \Delta_t - \inf_{t\in[0,1]} \Delta_t \quad \text{and} \quad \lim_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} = 2\,\mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t\right], \tag{21}$$

$$\mathbb{P}(\max_{k=1,\dots,n} Z_k \le -1) = \frac{1}{2}\mathbb{P}(T_0 > n) \sim \frac{a_n}{n} \max\left(1 - \frac{1}{2\alpha}, \frac{1}{2}\right) \mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t\right]. \tag{22}$$

Proof. Since $\mathbb{P}(\xi_1 \in \{-1, 0, 1\})$, $\mathcal{R}_n = \max_{k=0,\dots,n} Z_k - \min_{k=0,\dots,n} Z_k + 1$. Due to Proposition 13, we deduce the convergence in distribution of $(\mathcal{R}_n/a_n)_{n\in\mathbb{N}}$ and the convergence of $(\mathbb{E}[\mathcal{R}_n]/a_n)_{n\in\mathbb{N}}$. The last part of Theorem 16 follows from Theorem 7 since $(a_n)_{n\in\mathbb{N}}$ is γ-regular with $\gamma = \max\left(1 - \frac{1}{2\alpha}, \frac{1}{2}\right)$. □

Remark 17. *It is worth noticing that the techniques we used in this section can be applied to more general RWRS, for instance to RWRS studied in* [32].

5. Random process in Brownian scenery

Let us consider generalizations of the Kesten-Spitzer's process $(\Delta_t)_{t\ge0}$ introduced in the previous section. Let $W = \{W(x); x \in \mathbb{R}\}$ be a standard two-sided real Brownian motion and $Y = \{Y(t); t \ge 0\}$ be a real-valued self-similar process of index $\gamma \in (0, 2)$ with stationary increments. We assume that there exists a continuous version $\{L_t(x); x \in \mathbb{R}, t \ge 0\}$ of the local time of Y. The processes W and Y are defined on the same probability space and are assumed to be independent. We consider *the random process in Brownian scenery* $\{\Delta_t; t \ge 0\}$ defined as

$$\Delta_t = \int_{\mathbb{R}} L_t(x)\, dW(x).$$

The process Δ is itself a self-similar process of index h with stationary increments, with

$$h := 1 - \frac{\gamma}{2}.$$

Let $V_1 := \int L_1^2(x)\, dx$ be the self-intersection local time of Y. The following assumption is made on the random variable V_1.

(H1): There exist a real number $\alpha > 1$, and positive constants C, c such that for any $x \geq 0$,

$$\mathbb{P}\Big(V_1 \geq x\Big) \leq C \exp(-cx^{\alpha}).$$

In [12], examples of processes Y satisfying the above assumptions are given: stable Lévy process with index $\delta \in (1,2]$ (it satisfies (H1) with $\alpha = \delta$, see Lemma 2.2 in [12]), the fractional Brownian motion with index $H \in (0,1)$ (it satisfies (H1) with $\alpha = \frac{1}{H}$, see Lemma 2.3 in [12]) and the iterated Brownian motion which satisfies assumption (H1) with $\alpha = \frac{4}{3}$ (see Lemma 2.4 in [12]). Our main result is the following one.

Theorem 18. *Assume (H1). There exists a constant $c > 0$, such that for large enough T ,*

$$c^{-1}\, T^{-\frac{\gamma}{2}} (\ln T)^{-\frac{1}{2-\gamma}(1+\frac{1}{\alpha})} \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \Delta_t \leq 1\Big) \leq cT^{-\frac{\gamma}{2}}. \tag{23}$$

Conditionally to the process Y, $(\Delta_t)_{t\geq 0}$ is a centered Gaussian process with positively associated increments. Therefore, the discrete-time process $(\Delta_n)_{n\in\mathbb{N}}$ (with $\Delta_0 = 0$) satisfies the assumptions of Theorem 5 (note that $\mathbb{E}[\sup_{t\in[0,1]} \Delta_t]$ is finite using Lemma 3.3 and inequality (3.6) in [12]) then we deduce the upper bound

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \Delta_t \leq 1\Big) \leq \mathbb{P}\Big(\max_{k=1,\dots,\lfloor T\rfloor} \Delta_k \leq 1\Big) = \mathcal{O}(T^{-\frac{\gamma}{2}}).$$

Moreover, from Lemma 3.3 in [12], $(\Delta_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Theorem 10 with $b_n = c_0\ (\log n)^{(1+\alpha)/2\alpha}$ for a suitable c_0. Then,

$$\liminf_{n\to+\infty} n^{\frac{\gamma}{2}} (\ln n)^{\frac{1+\alpha}{2\alpha}} \mathbb{P}\Big(\max_{k=1,\dots,n} \Delta_k \leq 0\Big) > 0.$$

Let us prove the lower bound in Theorem 18. We follow the proof of Theorem 1 by remarking that from (3.2) and (3.6) in [12], there exists some constant $a > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} \Delta_t > x\right) = O\big(e^{-ax^{\frac{2\alpha}{1+\alpha}}}\big).$$

6. The Matheron-de Marsily model

Finally, we will consider particular models of random walks $(M_n)_{n\in\mathbb{N}}$ in random environment on $\mathbb{Z}^2$. We are namely interested in the survival probability of a particle evolving on a randomly oriented lattice introduced by Matheron and de Marsily in [26] (see also [5]) to model fluid transport in a porous stratified medium. Supported by physical arguments, numerical simulations and comparison with the fractional Brownian motion, Redner [28] and Majumdar [24] conjectured that the survival probability asymptotically behaves as $n^{-\frac{1}{4}}$. In this paper we rigorously prove their conjecture.

Let us describe more precisely the model and the results. Let us fix $p \in (0,1)$. The (random) environment will be given by a sequence $\xi = (\xi_k)_{k\in\mathbb{Z}}$ of i.i.d. centered random variables with values in $\{\pm 1\}$ and defined on the probability space $(\Omega, \mathcal{T}, \mathsf{P})$. Given ξ, the position of the particle M is defined as a $\mathbb{Z}^2$-random walk on nearest neighbours starting from 0 (i.e. $\mathbb{P}^{\xi}(M_0 = 0) = 1$) and with transition probabilities

$$\mathbb{P}^{\xi}(M_{n+1} = (x+\xi_y, y)|M_n = (x,y)) = p, \quad \mathbb{P}^{\xi}(M_{n+1} = (x, y\pm 1)|M_n = (x,y)) = \frac{1-p}{2}.$$

At site (x,y), the particle can either get down (or get up) with probability $\frac{1-p}{2}$ or move with probability p on the y's horizontal line according to its orientation (to the right (resp. to the left) if $\xi_y = +1$ (resp. if $\xi_y = -1$)). We will write $\mathbb{P}$ for the annealed law, that is the integration of the quenched distribution $\mathbb{P}^\xi$ with respect to P.

In the sequel, this random walk will be named MdM random walk. This 2-dimensional random walk in random environment was first studied rigorously [8]. They proved that the MdM random walk is transient under the annealed law $\mathbb{P}$ and under the quenched law $\mathbb{P}^\xi$ for P-almost every environment ξ. It was also proved that it has speed zero. Actually the MdM random walk is closely related to RWRS. This fact was first noticed in [15]. More precisely its first coordinate can be viewed as a generalized RWRS, the second coordinate being a lazy random walk on $\mathbb{Z}$ (see Section 5 of [10] for the details). Using this remark, a functional limit theorem was proved in [15] and a local limit theorem was established in [10], more precisely there exists some constant C only depending on p such that $\mathbb{P}(M_{2n} = (0,0)) \sim Cn^{-\frac{5}{4}}$. Since the random walk M *does not* have the Markov property under the annealed law, we are not able to deduce the survival probability from the previous local limit theorem. Let us define that the survival probability is the probability that the particle does not visit the $y-$axis (or the line $x = 0$) before time n i.e. $\mathbb{P}(T_0^{(1)} > n)$ where

$$T_0^{(1)} := \inf\{n \geq 1 \ : \ M_n^{(1)} = 0\}$$

is the first return time of the first coordinate $M^{(1)}$ of M to 0. Due to [15], the first coordinate $M^{(1)}_{\lfloor nt \rfloor}$ normalized by $n^{\frac{3}{4}}$ converges in distribution to $K_p\Delta_t^{(0)}$, where $K_p := \frac{p}{(1-p)^{\frac{1}{4}}}$ and where $\Delta^{(0)}$ is the Kesten-Spitzer process Δ with U and Y two independent standard Brownian motions. As for RWRS, the asymptotic behavior of this probability will be deduced from the range $\mathcal{R}_n^{(1)}$ of the first coordinate i.e. the number of vertical lines visited by $(M_k)_k$ up to time n, namely

$$\mathcal{R}_n^{(1)} := \#\{x \in \mathbb{Z} \ : \ \exists k = 0, ..., n, \ \exists y \in \mathbb{Z} \ : \ M_k = (x,y)\}.$$

The main result for the MdM random walk is the following.

Theorem 19.

$$\lim_{n\to+\infty} n^{-\frac{3}{4}} \mathbb{E}\left[\max_{k=0,\dots,n} M_k^{(1)}\right] = K_p \mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t^{(0)}\right].$$

The sequence $(\mathcal{R}_n^{(1)}/n^{\frac{3}{4}})_{n\in\mathbb{N}^*}$ *converges in distribution to* $K_p\left(\sup_{t\in[0,1]}\Delta_t^{(0)} - \inf_{t\in[0,1]}\Delta_t^{(0)}\right)$.
Moreover

$$\lim_{n\to+\infty} \frac{\mathbb{E}[\mathcal{R}_n^{(1)}]}{n^{\frac{3}{4}}} = 2K_p\,\mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t^{(0)}\right] \tag{24}$$

and

$$\lim_{n\to+\infty} n^{\frac{1}{4}} \mathbb{P}(T_0^{(1)} > n) = \frac{3}{2} K_p\,\mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t^{(0)}\right]. \tag{25}$$

Proof. Due to [15], $((M^{(1)}_{\lfloor nt \rfloor}/n^{\frac{3}{4}})_t)_{n\in\mathbb{N}^*}$ converges in distribution to $(K_p\Delta_t^{(0)})_t$ in the Skorohod space endowed with the J_1-metric. Hence $(n^{-\frac{3}{4}}(\max_{k=0,\dots,n} M_k^{(1)} - \min_{\ell=0,\dots,n} M_\ell^{(1)}))_{n\in\mathbb{N}^*}$ converges in distribution to $K_p(\sup_{t\in[0,1]}\Delta_t^{(0)} - \inf_{s\in[0,1]}\Delta_s^{(0)})$. Let us prove that this sequence is uniformly integrable. To this end we will prove that it is bounded in L^2.
Recall that the second coordinate of the MdM random walk is the lazy random walk. Let us denote it by $(S_n)_{n\in\mathbb{N}}$. Observe that

$$M_n^{(1)} := \sum_{k=1}^n \varepsilon_{S_k} \mathbf{1}_{\{S_k = S_{k-1}\}} = \sum_{y\in\mathbb{Z}} \varepsilon_y \tilde{N}_n(y),$$

with $\tilde{N}_n(y) := \#\{k = 1, ..., n \ : \ S_k = S_{k-1} = y\}$. Observe that $\tilde{N}$ is measurable with respect to the random walk S and that $0 \le \tilde{N}_n(y) \le N_n(y)$.
Conditionally to the walk S, the increments of $(M_n^{(1)})_{n\in\mathbb{N}}$ are centered and positively associated. It follows from Theorem 2.1 of [14] that

$$\mathbb{E}\left[\left|\max_{j=0,\dots,n} M_j^{(1)}\right|^2 \Big| S\right] \le c_2 \mathbb{E}\left[|M_n^{(1)}|^2 | S\right] \le c_2 \sum_{y\in\mathbb{Z}} (\tilde{N}_n(y))^2 \le c_2 V_n,$$

where again $V_n = \sum_{y\in\mathbb{Z}} (N_n(y))^2$. Therefore $\mathbb{E}\left[\left|\max_{j=0,\dots,n} M_j^{(1)}\right|^2\right] \le c_2 \mathbb{E}[V_n]$. Again, the result follows from the fact that $\mathbb{E}[V_n] \sim c' n^{\frac{3}{2}}$. Then, (25) directly follows from (4) or (5) in Theorem 7.

□

Remark 20. *In the historical model* [26]*, the probability p is equal to* $1/3$*, and in this particular case the survival probability is similar to $\kappa n^{-\frac{1}{4}}$ where $\kappa = \left(\frac{3}{2^5}\right)^{1/4} \mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t^{(0)}\right]$. An open question is to compute the value of the above expectation. Numerical simulations give* $\mathbb{E}\left[\sup_{t\in[0,1]} \Delta_t^{(0)}\right] \approx 0.54$.

Proof of Theorem 4. We conclude due to Theorem 19, since

$$\mathbb{P}\left(\max_{k=1,\dots,n} M_k^{(1)} \le -1\right) = \mathbb{P}\left(T_0^{(1)} > n, \ M_1 = (-1, 0)\right) = \frac{1}{2}\mathbb{P}(T_0^{(1)} > n).$$

□

Remark 21 (Range in the historical MdM model)**.** *It is worth noticing that the range $\mathcal{R}_n$ of the MdM random walk, i.e. the number of sites visited by M before time n, $\mathcal{R}_n := \#\{M_0, \dots, M_n\}$ is well understood. Using again* [34, Lemma 3.3.27]*, $(\mathcal{R}_n/n)_{n\in\mathbb{N}^*}$ converges $\mathbb{P}$-almost surely to* $\mathbb{P}[M_j \ne 0, \ \forall j \ge 1]$*, which contradicts the result announced in* [21].

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